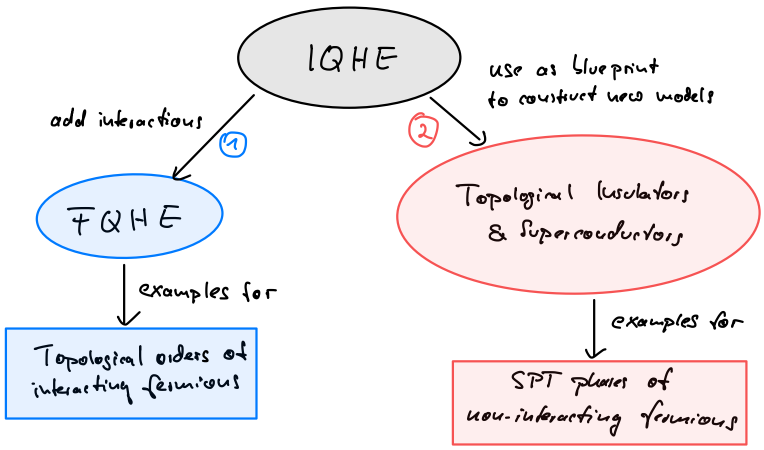
↓ **Lecture 8** [09.05.25]

1.7. Notes on classification

The IQHE is an important corner stone in the theory of topological phases, both historically and conceptually. Starting from the IQHE, there are (at least) two directions to explore:



(1) Keep the QHE setting but consider fractionally filled Landau levels:

→ Interactions become important (flat bands!)

→ ↑ *Fractional quantum Hall Effect* (FQHE)

→ States with ← *(non-invertible) topological order*
with anyonic excitations and fractional charges (depending on the filling)

(2) Leave the QHE setting but stay in the realm of *non-interacting fermions* (on the lattice):

→ Construct lattice models with topological bands …

- …w/o magnetic fields (?)
- …w/o breaking time reversal symmetry (?)
- …w/o particle-number conservation (?)

→ → *Topological insulators & superconductors*
[SPT phases of non-interacting fermions & invertible topological orders]

In the following we will pursue **Path 2** which will eventually lead us to the "periodic table of topological insulators and superconductors" in Chapter 6.

Note:

IQH states (= filled Landau levels) are part of the classification of topological phases of non-interacting fermions that we will introduce [92]. However, they are also long-range entangled [35], but this long-range entanglement is of a special "boring" kind in that it does not give rise to fancy anyonic statistics of excitations. In our nomenclature, IQH states are examples of ← *invertible topological order*. (You can locally disentangle a IQH state by "gluing" a time-inverted copy on top.)

According to another naming scheme [different from the one I introduced], IQH states are "short-range entangled" because they lack anyonic excitations and their → *topological entanglement entropy* vanishes

[36, 37]. Because of the time-reversal symmetry breaking and the chiral nature of their edge modes, some call IQH states simply *chiral phases* [93, 94].

It is noteworthy that symmetry *does* play a role for the IQH, namely the U(1) symmetry that describes the *conservation of charge*. It does neither protect the entanglement structure nor the chiral edge states, but it is necessary for the quantization of the Hall response [35, 93, 94]. (Which makes sense: in a material where charge can randomly enter or leave the sample, there is no reason for a conductivity to be quantized.)

2. Topological Bands without Magnetic Fields: The Quantum Anomalous Hall Effect

2.1. Preliminaries

We seek for models with the following properties:

- Lattice model (of non-interacting fermions)
- Band insulator
- Non-zero Chern number
- No magnetic field (!)

The first three conditions are satisfied by the ↑ *Hofstadter model*, a lattice model that captures the IQHE physics. However, the Hofstadter model is a rather complicated multiband model due to the enlarged magnetic unit cell (➔ Problemset 4). This motivates the question:

> *Are there models without external magnetic field that have Chern bands?*

- Chern band = Band with non-zero Chern number
- Note that the sum of Chern numbers of all bands is always zero (➔ Problemset 3). Thus, if the answer to this question is affirmative, the model must have at least two bands. This can be achieved either with an internal degree of freedom (spin) or, alternatively, with sublattice degrees of freedom (i.e., a unit cell with more than one site).

Before we proceed, let us fix the nomenclature:

⁂ Definition: Chern insulator

$$\text{Chern insulator* (CI*)} \;:=\; \begin{cases} \text{Lattice model} \\ \text{Band insulator} \\ \text{Non-zero Chern number} \end{cases} \tag{2.1}$$

Prototype: ↑ *Hofstadter model*

$$\text{Chern insulator (CI)} \;:=\; \begin{cases} \text{Lattice model} \\ \text{Band insulator} \\ \text{Non-zero Chern number} \\ \text{No external magnetic field} \end{cases} \tag{2.2}$$

Prototype: → *Haldane model*

With this definition, the above question can be restated:

Are there Chern insulators?

Before we focus on specific models, let us explore some generic properties of translation invariant models with two bands:

2.1.1. Lattice models with two bands

1 | General setting: (The following is crucial throughout Part I!)

We start with a brief review of Hamiltonians that describe non-interacting fermions in translation invariant lattice models (here with any number of bands in any dimension):

i | ⊲ Single-particle (SP) Hilbert space $\mathcal{H} = \operatorname{span}\{|\Psi_{i\alpha}\rangle\}_{i\alpha}$ with SP Hamiltonian

$$H = \sum_{i\alpha,j\beta} H_{i\alpha,j\beta}|\Psi_{i\alpha}\rangle\langle\Psi_{j\beta}| \tag{2.3}$$

$i = 1\ldots N$: site index
$\alpha = 1\ldots M$: internal degrees of freedom (e.g. multiple sites per unit cell, spin, …)

ii | → Many-body (MB) Hilbert space $\hat{\mathcal{H}} = \bigoplus_n \bigwedge^n(\mathcal{H})$ with MB Hamiltonian

$\hat{\mathcal{H}}$ is the fermionic ↓ *Fock space* (the ↑ *exterior algebra* of $\mathcal{H}$); $\bigwedge^n$ denotes the nth ↑ *exterior power* of the single-particle Hilbert space $\mathcal{H}$.

$$\hat{H} = \sum_{i\alpha,j\beta} c^\dagger_{i\alpha} H_{i\alpha,j\beta} c_{j\beta} \tag{2.4}$$

$c^\dagger_{i\alpha}/c_{i\alpha}$: fermionic creation/annihilation operators for fermion in state $|\Psi_{i\alpha}\rangle$

The fact that this Hamiltonian only includes *quadratic* terms of fermionic operators makes it exactly solvable; one says that $\hat{H}$ describes *quadratic fermions*, *non-interacting fermions*, or *free fermions*.

iii | Assume Translation symmetry $\overset{\circ}{\rightarrow}$

$$\hat{H} = \sum_{\boldsymbol{k};\alpha,\beta} c^\dagger_{\boldsymbol{k}\alpha} H_{\alpha\beta}(\boldsymbol{k}) c_{\boldsymbol{k}\beta} \tag{2.5}$$

with ⁂ *momentum modes*

$$c_{\boldsymbol{k}\alpha} := \frac{1}{\sqrt{N}} \sum_i e^{i\boldsymbol{x}_i\boldsymbol{k}} c_{i\alpha} \tag{2.6}$$

$\boldsymbol{x}_i$: position of site i

iv | So the SP Hamiltonian decomposes as $H = \bigoplus_{\boldsymbol{k}} H(\boldsymbol{k})$ with ⁂ *Bloch Hamiltonian* $H(\boldsymbol{k})$ (a Hermitian $M \times M$-matrix).

Diagonalizing the latter yields

$$H(\boldsymbol{k}) = \sum_n E_n(\boldsymbol{k})|u_{n\boldsymbol{k}}\rangle\langle u_{n\boldsymbol{k}}| \tag{2.7}$$

- $|u_{\boldsymbol{k}n}\rangle$: Bloch wavefunction
- $n = 1 \dots M$: band index
- $E_n(\boldsymbol{k})$: SP spectrum

$\rightarrow$ The SP Hilbert space decomposes as $\mathcal{H} = \bigoplus_{\boldsymbol{k}} \mathcal{H}_{\boldsymbol{k}}$ with momentum mode space $\mathcal{H}_{\boldsymbol{k}} = \text{span}\{|u_{n\boldsymbol{k}}\rangle\}_n$.

2 | We now specialize to models with *two* bands on a 2D lattice …

$\triangleleft$ Most general two-band Hamiltonian on a 2D lattice:

$$H = \bigoplus_{\boldsymbol{k}\in T^2} H(\boldsymbol{k}) \quad \text{with} \quad H(\boldsymbol{k}) = \varepsilon(\boldsymbol{k})\,\mathbb{1} + \vec{d}(\boldsymbol{k})\cdot\vec{\sigma} \tag{2.8}$$

- T^2: Brillouin zone (BZ) (= Torus)
- σ^α with $\alpha = x, y, z$: Pauli matrices
- $\vec{d}(\boldsymbol{k}) : T^2 \rightarrow \mathbb{R}^3$: real, vector-valued function on BZ

The two spatial dimensions are responsible for the Brillouin zone being a 2-torus T^2, the two bands allow us to expand the Hermitian 2×2-matrix $H(\boldsymbol{k})$ into Pauli matrices.

3 | Spectrum:

$$E_\pm(\boldsymbol{k}) \mathrel{\hat{=}} \varepsilon(\boldsymbol{k}) \pm |\vec{d}(\boldsymbol{k})| \tag{2.9}$$

$\rightarrow$ Band insulator iff

$$\min_{\boldsymbol{k}\in T^2} E_+(\boldsymbol{k}) > \max_{\boldsymbol{k}\in T^2} E_-(\boldsymbol{k}) \tag{2.10}$$

Strictly speaking, this condition *allows* the system to be a band insulator *if* the chemical potential (= Fermi energy E_F) is in the gap (which the above condition guarantees to exist). We assume this situation in the following: $\min_{\boldsymbol{k}\in T^2} E_+(\boldsymbol{k}) > E_F > \max_{\boldsymbol{k}\in T^2} E_-(\boldsymbol{k})$.

$\triangleleft$ Weaker condition:

$$\forall \boldsymbol{k} \in T^2 : \quad E_+(\boldsymbol{k}) - E_-(\boldsymbol{k}) = 2|\vec{d}(\boldsymbol{k})| > 0 \tag{2.11}$$

This means that the two bands never touch and/or intersect.

$\rightarrow$ Normalization possible:

$$\hat{d}(\boldsymbol{k}) := \frac{\vec{d}(\boldsymbol{k})}{|\vec{d}(\boldsymbol{k})|} \quad \text{such that} \quad \hat{d} : T^2 \rightarrow S^2 \tag{2.12}$$

S^2: unit sphere in $\mathbb{R}^3$

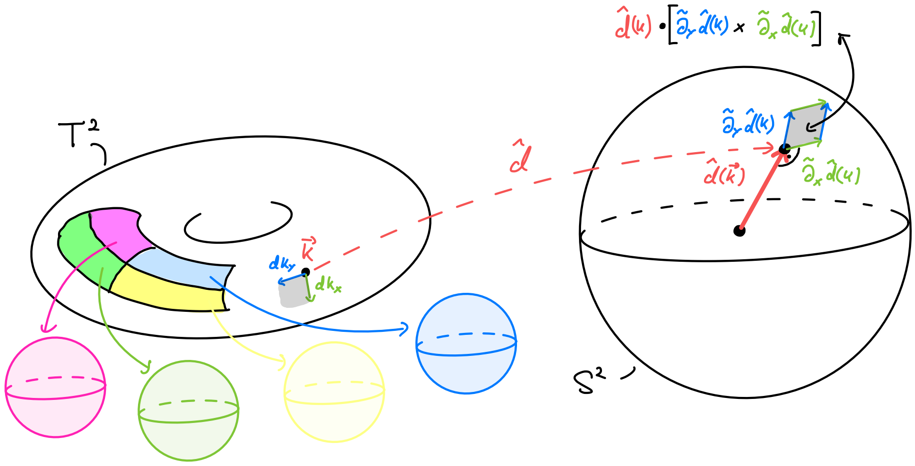
4 | Chern number of the lower band:

$$C \mathrel{\hat{=}} \underbrace{-\frac{1}{4\pi}\int_{T^2} \overbrace{\hat{d}(\boldsymbol{k})\cdot[\tilde{\partial}_x\hat{d}(\boldsymbol{k})\times\tilde{\partial}_y\hat{d}(\boldsymbol{k})]}^{\substack{\text{(Oriented) Jacobian for surface integral} \\ \propto\ \text{Berry curvature}}}\,\mathrm{d}^2k}_{4\pi\mathbb{Z}} \in \mathbb{Z} \tag{2.13}$$

Derivation: ➔ Problemset 5

5 | Geometric interpretation:

The expression for the Berry curvature is just the ↓ *Jacobian* for the (oriented) surface integral over the sphere S^2:



i | → C counts how often $\hat{d}(\boldsymbol{k})$ covers S^2 when sweeping over the Brillouin zone T^2

Note that this can only happen in integer steps since the area element in Eq. (2.13) is *oriented*: "going back" counts negative.

ii | $C = C[\hat{d}] \in \mathbb{Z}$ is a topological invariant

This implies in particular that two different maps $\hat{d}_a$ and $\hat{d}_b$ that can be continuously deformed into each other must have the same winding number C.

Mathematically this follows because Eq. (2.13) is a continuous function of $\hat{d}$ and maps into the integers. It is a well-known fact from topology that such functions are constant on their domain.

iii | Hamiltonian H_a can be continuously deformed into H_b without closing the gap

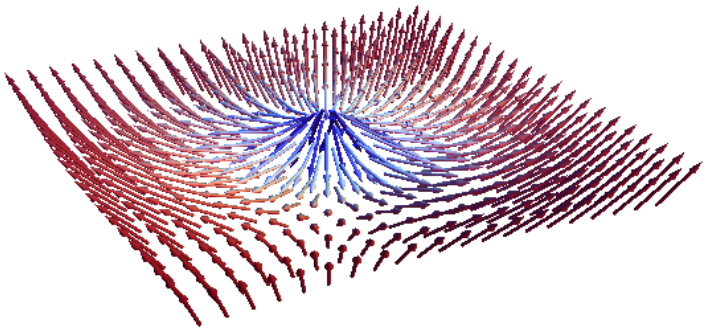
⇔ $\hat{d}_a$ can be continuously deformed into $\hat{d}_b$

Note that when the gap closes, the normalized vector $\hat{d}$ has a *singularity* (= is undefined) somewhere on T^2 so that Eq. (2.13) is undefined as well.

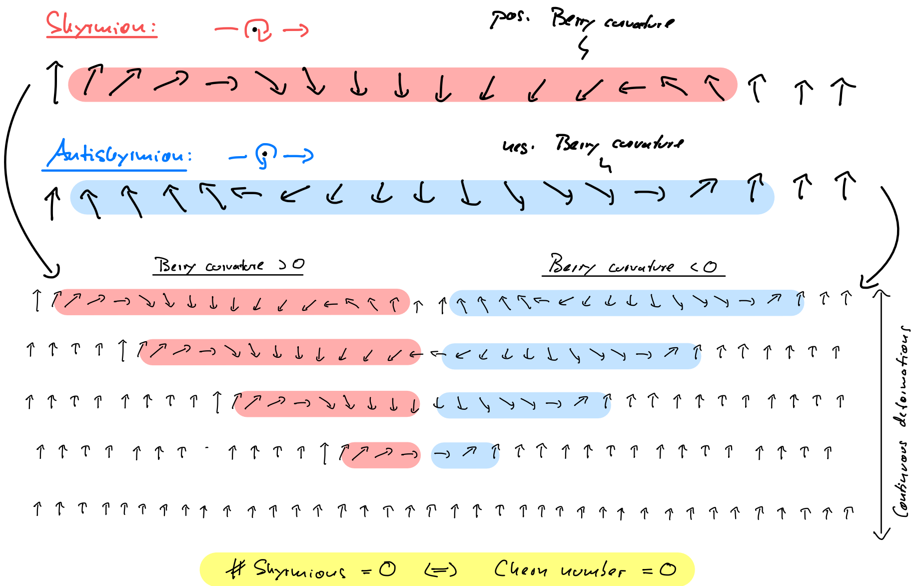
iv | → C labels different topological phases

6 | ‡ Skyrmion interpretation:

i | The region on T^2 where the field $\hat{d}(\boldsymbol{k})$ "wraps around the sphere" can be quite localized. This creates a local "knot" in the field that can be viewed as an excitation of a specific type of non-linear field theory known as ↑ *non-linear sigma models*. In this (very different) context, these localized excitations are called ↑ *skyrmions* (after Tony Skyrme who introduced them to describe the strong force [95]); they are an example for ↑ *topological solitons*. Here an illustration of a skyrmion that represents a field $\hat{d}$ wrapping once around the sphere:



ii | If the direction how the field sweeps over the sphere is inverted, one ends up with an *anti-skyrmion*. A single skyrmion is a topologically protected field configuration and cannot be removed by continuous deformations of $\hat{d}$ (this is just our argument from above about the topological character of C restated in terms of skyrmions). However, a skyrmion and an antiskyrmion *can* be continuously removed (they "annihilate" each other):



(This is a 1D cut through the 2D surface on which the skyrmion-antiskyrmion pair lives.)

iii | Summary:

- Skyrmions are "twists" of $\hat{d}$ and "live" on the BZ
- Positive (negative) Berry curvature indicates a finite (anti-)skyrmion *density*
- The Chern number is the number of skyrmions minus the number of antiskyrmions

iv | An interesting mathematical tangent:

† Note: Pontryagin number

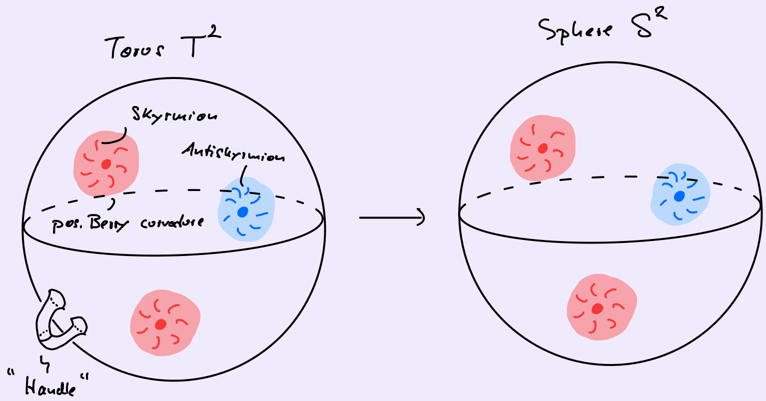
The fact that $\hat{d}$ lives on a torus T^2 (the Brillouin zone) is not important in this situation. Thus it is possible to replace the torus T^2 by a sphere S^2 (which can be seen as the one-point compactified momentum space $\mathbb{R}^2$ of the continuum). Then

$$\hat{d} : S^2 \to S^2 \tag{2.14}$$

is a *continuous* function that maps the sphere onto the sphere. Two Hamiltonians H_a and H_b belong to the same phase, if the corresponding functions $\hat{d}_a$ and $\hat{d}_b$ can be "smoothly deformed" into each other.

In topology, such a smooth deformation of one function into another is known as a ↑ *homotopy*; the set of equivalence classes under homotopy has a group structure and is known as (second) homotopy group of S^2, write $\pi_2(S^2)$; it is well-known that $\pi_2(S^2) = \mathbb{Z}$. The equivalence classes in $\pi_2(S^2)$ can be labeled by an integer known as ↑ *Pontryagin number*; it counts how often a map $\hat{d}$ traces out the (target) sphere S^2 when sweeping the (domain) sphere S^2. In the current situation, this is exactly the Chern number C.

That the torus can be replaced by a sphere is also evident in the skyrmion picture. Since the skyrmions can be localized, they do not care whether they live on a torus or a sphere:



However, note that $\hat{d}$ can have "twists" around the torus that are not reflected in the Chern number (and are not related to skyrmions). These "twists" give rise to ↑ *weak topological indices* which can have physical effects on the boundary physics in specific directions [57, 96, 97]. Since these effects rely on the domain of $\hat{d}$ to be a torus (= Brillouin zone), they are protected by the translation symmetry of the lattice (this makes them "weak"). Weak topological indices are not important for the models discussed below.

Institute for Theoretical Physics

2.1.2. Time-reversal symmetry (TRS)

¡! We will introduce time-reversal symmetry as the first of three "generic" symmetries and discuss the restrictions it imposes on the Bloch Hamiltonian $H(\boldsymbol{k})$. It plays a role for the → *Haldane model* but *not* as protecting symmetry; quite the contrary: it must be *broken* to make the model interesting (recall that the IQHE – which we would like to mimic – is not an SPT phase). However, in upcoming lectures (throughout Part I) we will use this symmetry as a *protecting symmetry* instead, which then leads us to the concept of → *topological insulators* and their classification.

1 | $\sphericalangle$ Single particle with SP Hilbert space $\mathcal{H}$:

TRS $T\,:\,t\mapsto -t$ is a $\mathbb{Z}_2$-symmetry (inverting time twice should do nothing!) and sould reasonably act as

$$TxT^{-1}\stackrel{!}{=}x \quad \text{but} \quad TpT^{-1}\stackrel{!}{=}-p \tag{2.15}$$

$\rightarrow Ti\hbar T^{-1}=T[x,p]T^{-1}=-[x,p]=-i\hbar$
$\rightarrow T$ must be antiunitary:

$$T_U=U\mathcal{K} \quad \text{with} \quad \mathcal{K}=\text{Complex conjugation} \tag{2.16}$$

U: unitary operator that determines the *representation* T_U of T on the SP Hilbert space

↑ *Wigner's theorem* [98] states that a *symmetry* (i.e. an operator O that preserves all probability amplitudes, $|\langle O\Psi|O\Phi\rangle|^2=|\langle\Psi|\Phi\rangle|^2$) acts either as a *unitary* or an *antiunitary* operator on the Hilbert space (➲ Problemset 1). In combination with $TiT^{-1}=-i$, this fixes T to the generic form T_U above.

$\rightarrow$ SP Hamiltonian H is ⁂ *time-reversal symmetric* iff $[H,T_U]=0$ for a U chosen appropriately to describe the system (→ *below*)

Explicitly the condition for time reversal symmetry reads:

$$HU\mathcal{K}=HT_U=T_UH=U\mathcal{K}H \;\Leftrightarrow\; HU=UH^* \;\Leftrightarrow\; H=UH^*U^\dagger \tag{2.17}$$

2 | T_U is antiunitary →

$$T_U^2=UU^*=U(U^T)^{-1} \tag{2.18}$$

T_U is projective representation of $\mathbb{Z}_2$ →

$$T_U^2=\lambda\mathbb{1} \quad \text{with} \quad |\lambda|=1 \tag{2.19}$$

- Being a → *projective representation* of $\mathbb{Z}_2$ realizes our notion that inverting time twice should bring us back to the same physical state: Because physical states are *rays* (➲ Problemset 1), this only means that T_U^2 applied to a state vector gives back the same vector *up to a phase*. This phase must be the same for all states since otherwise you could superimpose two states with different phases to construct a state that transforms to a physically distinct state under T_U^2 – in contradiction with our assumption that inverting time twice has no physical consequences.
- Here is an alternative, more generic line of arguments that does not require the assumption that time reversal is a $\mathbb{Z}_2$ symmetry as input [92]:

  Assume that you made the total Hamiltonian block-diagonal by "using up" all its potential unitary symmetries. Then *each block* carries an irreducible representation of the unitary symmetry group and an "irreducible Hamiltonian" so that the arguments below hold. How T is represented in each block can vary, however the result for T^2 must be the same on all blocks because otherwise T is not a (projective) representation of $\mathbb{Z}_2$ on the whole SP Hilbert space—and this would contradict our intuition that applying time-reversal twice does nothing.

Now go back to Eq. (2.18) and note that ...

- UU^* is unitary
- $[H, UU^*] = 0 \to UU^*$ is a symmetry of H

$\lessdot$ Generic H without any additional unitary symmetries
$\to$ Hamiltonian irreducible
$\to T_U^2 = UU^* = \lambda\mathbb{1}$
(This is an application of ↑ *Schur's lemma* on the irreducible *Hamiltonian*.)

$$\text{Eqs. (2.18) and (2.19)} \quad \Rightarrow \quad U = \lambda U^T \overset{\bullet^T}{\Longleftrightarrow} U^T = U\lambda \tag{2.20a}$$

$$\Rightarrow \quad U = \lambda^2 U \tag{2.20b}$$

$$\Rightarrow \quad \lambda = \pm 1 \tag{2.20c}$$

$\to$

$$\boxed{T_U^2 = \pm\mathbb{1}} \tag{2.21}$$

If $T_U^2 = -\mathbb{1}$, T_U is an *antiunitary*, *projective* representation of $\mathbb{Z}_2$.

3 | Examples:

- $\lessdot$ Spinless particles: (=no internal degrees of freedom)

$$\boxed{T_0 := \underbrace{\mathbb{1}}_{U_0}\mathcal{K} \quad \Rightarrow \quad T_0^2 = +\mathbb{1}} \tag{2.22}$$

- $\lessdot$ Spin-$\frac{1}{2}$ particles with spin operator $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

Just as time reversal inverts the linear momentum p, it should also invert (internal) angular momentum (= spin):

$$T_U \vec{S} T_U^{-1} \overset{!}{=} -\vec{S} \tag{2.23}$$

So we want that $T_U \sigma^i T_U^{-1} \overset{!}{=} -\sigma^i$ for all Pauli matrices $i = x, y, z$.

Note that this choice is not arbitrary. For example, it is inconsistent to demand (nonzero) spin to be *invariant* under time-reversal ($T_U S_i T_U^{-1} = S_i$) because then $[S_i, S_j] = i\epsilon_{ijk} S_k$ (which defines spin operators) implies $[S_i, S_j] = -i\epsilon_{ijk} S_k$ (since T_U is still antiunitary) such that $S_k = 0$ ↯.

$\to$ Solution:

$$\boxed{T_{\frac{1}{2}} := \underbrace{\sigma^y}_{U_{\frac{1}{2}}}\mathcal{K} \quad \Rightarrow \quad T_{\frac{1}{2}}^2 = -\mathbb{1}} \tag{2.24}$$

- Note how $T_U \sigma^i T_U^{-1} = -\sigma^i$ is satisfied: for $i = y$ it follows from the complex conjugation $\mathcal{K}$ (antiunitarity), but for $i = x, z$ it follows because σ^x and σ^z are *real* matrices that anticommute with σ^y.
- The statement $T_U^2 = -\mathbb{1}$ is true for all particles with *half-integer* spin (but with other choices for U that depend on the spin, of course).
- Often you will find the choice $T_{\frac{1}{2}} = -i\sigma^y \mathcal{K}$. This follows if one derives $T_{\frac{1}{2}}$ as a spin rotation. Note that you can multiply $T_{\frac{1}{2}}$ with an arbitrary phase without changing its algebraic properties.

4 | Consequence of $T_U^2 = -\mathbb{1}$:

> **¡! Important: Kramers theorem**
>
> Every eigenenergy of a time-reversal invariant Hamiltonian H with $T_U^2 = -\mathbb{1}$ is at least two-fold degenerate.

Proof: ➔ Problemset 5

The theorem was discovered by Hans Kramers in 1930 and mathematically studied on general grounds by Eugene Wigner in 1932 [99]. It has far-reaching consequences: For instance, the degeneracy of atomic energy levels with half-integer total angular momentum cannot be lifted completely by electric fields alone (which preserve TRS); instead, magnetic fields are needed (which break TRS). → *Later* we will see that Kramers theorem restricts the band structure of time-reversal invariant systems in that it requires crossing bands at so called → *time-reversal invariant momenta* (TRIMs) in the Brillouin zone.