↓ **Lecture 6** [02.05.25]

5 | ⊲ Special case: Coupling to uniform electric field $\boldsymbol{E}(t) = \boldsymbol{E}e^{-i\omega t}$

i | Choose gauge such that $\boldsymbol{E}(t) = -\partial_t \boldsymbol{A}(t)$ (i.e. $A_t = \phi = \text{const}$)

Remember that in general $\boldsymbol{E} = -\nabla\phi - \partial_t \boldsymbol{A}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

$\rightarrow \boldsymbol{A}(t) = \boldsymbol{E}e^{-i\omega t}/(i\omega)$

ii | ⊲ Perturbation Hamiltonian:

$$\Delta H_I(t) = -\boldsymbol{J}(t) \cdot \boldsymbol{A}(t) \tag{1.63}$$

with (total) current operator $\boldsymbol{J}(t)$

- At this point we do not want to fix the unperturbed Hamiltonian H_0 that describes the charge carriers without the field. Hence we do not know the form of $\boldsymbol{J}(t)$ in the interaction picture. We therefore play it safe and carry a potential time-dependence along.
- This is a linearized version of the true coupling Hamiltonian that describes the effect of the electromagnetic field on electrical charges. For instance, a free particle with charge q (and with $\phi = \text{const} = 0$) is described by the Hamiltonian

$$H = \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2 = \underbrace{\frac{\boldsymbol{p}^2}{2m}}_{\sim H_0} \underbrace{- \overbrace{\frac{q\boldsymbol{p}}{m}}^{\sim \boldsymbol{J}} \cdot \boldsymbol{A}}_{\sim \Delta H(t)} + \cancel{\mathcal{O}(\boldsymbol{A}^2)}. \tag{1.64}$$

There is also a quadratic term $\boldsymbol{A}^2$ which does not contribute to the Hall conductance (so we can safely drop it).

- In therms of the ↓ *current density* $\boldsymbol{j}(\boldsymbol{r}, t)$ the Hamiltonian reads

$$\Delta H_I(t) = -\int \mathrm{d}^2 r\, \boldsymbol{j}(\boldsymbol{r}, t) \cdot \boldsymbol{A}(\boldsymbol{r}, t) \tag{1.65}$$

with the usual current density $\boldsymbol{j} = \frac{q}{2m}\sum_i [\boldsymbol{p}_i \delta(\boldsymbol{r} - \boldsymbol{r}_i) + \delta(\boldsymbol{r} - \boldsymbol{r}_i)\boldsymbol{p}_i]$ for many particles indexed by i. With a homogeneous electric field, this becomes

$$\Delta H_I(t) = -\boldsymbol{J}(t) \cdot \boldsymbol{A}(t) \quad \text{with total current} \quad \boldsymbol{J}(t) = \int \mathrm{d}^2 r\, \boldsymbol{j}(\boldsymbol{r}, t). \tag{1.66}$$

For a homogeneous current, the total current is $\boldsymbol{J} = L_x L_y \boldsymbol{j} = A\boldsymbol{j}$ where $A = L_x L_y$ denotes the area of the sample.

iii | ⊲ Current as observable: $\mathcal{O} = J_i \rightarrow$

(Remember that we set the static expectation value to zero: $\langle 0|J_i|0\rangle = 0$.)

$$\langle J_i(t)\rangle \overset{1.62}{=} -\frac{1}{\hbar\omega}\int_{-\infty}^{t} \langle 0| \left[J_j(t'), J_i(t)\right] |0\rangle E_j e^{-i\omega t'} \mathrm{d}t' \tag{1.67a}$$

Time-translation invariance of H_0; Substitution $t'' = t - t'$

$$\hat{=} \underbrace{\left\{-\frac{1}{\hbar\omega}\int_0^{\infty} \langle 0| \left[J_j(0), J_i(t'')\right] |0\rangle e^{i\omega t''} \mathrm{d}t''\right\}}_{=:\sigma_{ij}(\omega)A} E_j e^{-i\omega t} \tag{1.67b}$$

with ⁂ *conductivity tensor* $\sigma_{ij}(\omega)$

The sample area $A = L_x L_y$ shows up because the conductivity tensor relates, by definition, the current *density* j_i to the electric field, and not the total current $J_i = A j_i$.

To show the second equality, use that $J_j(t') = e^{\frac{i}{\hbar}H_0 t'} J_j e^{-\frac{i}{\hbar}H_0 t'}$ [and similar for $J_i(t)$] and that $|0\rangle$ is an eigenstate of H_0.

iv | → Hall conductivity:

$$\sigma_{xy}(\omega) = -\frac{1}{\hbar\omega A}\int_0^\infty \langle 0|\left[J_y(0), J_x(t)\right]|0\rangle\, e^{i\omega t}\mathrm{d}t \tag{1.68}$$

This is the *AC Hall conductivity* as it is still frequency dependent.

v | Set $t_0 = 0$ and use $U_0(t) = \sum_n e^{-iE_n t/\hbar}|n\rangle\langle n|$ and $J_i(t) = U_0^\dagger(t) J_i U_0(t)$:

→

$$\sigma_{xy}(\omega) = -\frac{1}{\hbar\omega A}\int_0^\infty \sum_n \begin{Bmatrix} \langle 0|J_y|n\rangle\langle n|J_x|0\rangle e^{i(E_n-E_0)t/\hbar} \\ -\langle 0|J_x|n\rangle\langle n|J_y|0\rangle e^{i(E_0-E_n)t/\hbar}\end{Bmatrix} e^{i\omega t}\mathrm{d}t \tag{1.69a}$$

Integrate (using a regularization $\omega + i\varepsilon$ to make the integral convergent)

$$= -\frac{i}{\omega A}\sum_{n\neq 0}\left\{\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle}{\hbar\omega + E_n - E_0} - \frac{\langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{\hbar\omega + E_0 - E_n}\right\} \tag{1.69b}$$

vi | Take DC limit $\omega \to 0$ and use $\frac{1}{\hbar\omega+E_n-E_0} = \frac{1}{E_n-E_0} - \frac{\hbar\omega}{(E_n-E_0)^2} + \mathcal{O}(\omega^2)$:

(Note the i/ω that must be canceled to render the expression finite!)

$$\boxed{\sigma_{xy} \overset{\circ}{=} \frac{i\hbar}{A}\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle - \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{(E_n - E_0)^2}} \tag{1.70}$$

This is the Hall conductivity expressed in terms of current matrix elements. Our → *next* project will be a (quite tedious) reformulation of this expansion with the goal to re-express it in terms of a topological invariant, namely the ← *Chern number*.

vii | Comment on the constant term:

For the derivation of Eq. (1.70) it is crucial that

$$\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle + \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{E_n - E_0} = 0 \tag{1.71}$$

which makes the constant terms of the Taylor expansion cancel (this avoids the divergence for $\omega \to 0$!).

One way to see this is from *rotation invariance* of the system in the x-y-plane (a quantum Hall system should be rotation invariant about the axis of the magnetic field). In particular, σ_{xy} should be invariant under the $\pi/2$-rotation $J_x \mapsto J_y$ and $J_y \mapsto -J_x$ (note that $\boldsymbol{J}$ is a vector operator). This means that

$$\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle + \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{E_n - E_0} \overset{!}{=} -\sum_{n\neq 0}\frac{\langle 0|J_x|n\rangle\langle n|J_y|0\rangle + \langle 0|J_y|n\rangle\langle n|J_x|0\rangle}{E_n - E_0} \tag{1.72}$$

which implies Eq. (1.71) so that only the *antisymmetric* part of σ_{xy} survives.

Note that this is a quite general argument: If we decompose the 2D conductivity tensor into symmetric and antisymmetric parts, $\sigma = \sigma_s + \sigma_a$, and demand rotational invariance of the tensor, i.e., $\sigma = R\sigma R^T$ for a 2D rotation matrix R, we have $\sigma_s = R\sigma_s R^T$ and $\sigma_a = R\sigma_a R^T$ separately. The only *symmetric* matrix invariant under rotations is proportional to the identity, $\sigma_s = \sigma_{xx} \cdot \mathbb{1}$, so that there cannot be a symmetric contribution to the off-diagonals (that is, the Hall conductivity σ_{xy}). Thus the most general form of a *rotation invariant* conductivity tensor is

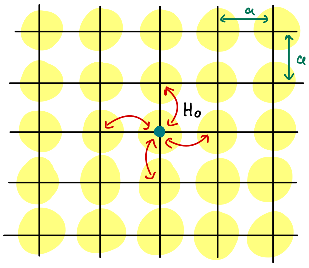
$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} . \tag{1.73}$$

1.4.2. The TKNN invariant

Here we want to connect the Hall conductivity [given by the Kubo formula Eq. (1.70)] to the Chern number and thereby explain the quantization of the former. To do so, we consider non-interacting electrons in a two-dimensional periodic potential, so that the momentum space is a torus.

The rationale of the following discussion is similar to the original approach by Thouless *et al.* [17].

1 | $\sphericalangle$ Single electron in a periodic potential with Hamiltonian H_0:



System size: $L_x \times L_y$ & periodic boundaries

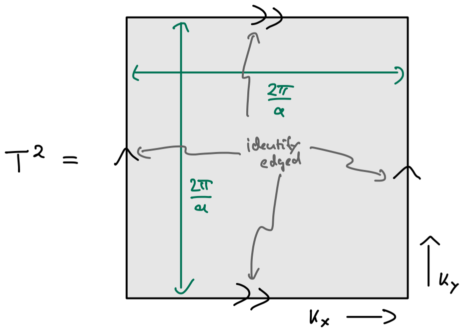
We take the thermodynamic limit $L_x, L_y \to \infty$ later.

2 | ↓ <u>*Bloch theorem*:</u>

- Eigenfunctions: $\Psi_{n\boldsymbol{k}} = e^{i\boldsymbol{k}\boldsymbol{x}} u_{n\boldsymbol{k}}(\boldsymbol{x})$
  with $u_{n\boldsymbol{k}}(\boldsymbol{x} + \boldsymbol{R}) = u_{n\boldsymbol{k}}(\boldsymbol{x})$ for lattice vectors $\boldsymbol{R}$ and band index $n = 1, 2, \dots$
- Eigenenergies $\varepsilon_n(\boldsymbol{k})$ continuous in $\boldsymbol{k}$ → "Bands"
- $\Psi_{n\boldsymbol{k}+\boldsymbol{K}} = \Psi_{n\boldsymbol{k}}$ for reciprocal lattice vectors $\boldsymbol{K}$

  If $\boldsymbol{R} = an_x\boldsymbol{e}_x + an_y\boldsymbol{e}_y$ describes a square lattice with lattice constant a, the reciprocal lattice is $\boldsymbol{K} = m_1\boldsymbol{k}_1 + m_2\boldsymbol{k}_2$ with $\boldsymbol{k}_i = \frac{2\pi}{a}\boldsymbol{e}_i$.

  → Brillouin zone = Torus T^2



Since our system is finite, momenta are discrete. The size of the Brillouin zone is determined by the inverse lattice constant and remains fixed in the following.

3 | $\lessdot$ Many-body Fock states with Fermi energy E_F:

¡! While we can understand the integer quantum Hall effect within the framework of non-interacting fermions, the quantization of the Hall conductivity is a genuine quantum many-body phenomenon. It is crucial that you understand the difference (and relation) between these concepts.

$$\text{Ground state} = |0\rangle \mapsto |\mathbf{0}\rangle = \text{Filled Fermi sea} \tag{1.74a}$$

$$\text{Excited states} = |n\rangle \mapsto |\boldsymbol{n}\rangle = \text{Fermi sea with particle-hole excitations} \tag{1.74b}$$

$$\text{Current operator} = J_i \mapsto \mathfrak{J}_i = \text{Second-quantized current operator} \tag{1.74c}$$

In the following, **bold states** live in the fermionic Fock space (= many-body states), whereas states in normal font live in the single-particle Hilbert space.

4 | Eq. (1.70) → Hall conductivity of fermionic many-body system:

$$\sigma_{xy} \stackrel{\circ}{=} \frac{i\hbar}{A} \sum_{\boldsymbol{n}\neq\mathbf{0}} \frac{\langle\mathbf{0}|\mathfrak{J}_y|\boldsymbol{n}\rangle\langle\boldsymbol{n}|\mathfrak{J}_x|\mathbf{0}\rangle - \langle\mathbf{0}|\mathfrak{J}_x|\boldsymbol{n}\rangle\langle\boldsymbol{n}|\mathfrak{J}_y|\mathbf{0}\rangle}{(E_{\boldsymbol{n}} - E_{\mathbf{0}})^2} \tag{1.75}$$

Note that the sum goes over all possible excited many-body states (which are all states except the Fermi sea ground state). However, below we will see that only states with a single particle-hole excitation contribute.

5 | Current operator = Single-particle operator:

$$\mathfrak{J}_i = \sum_{n\boldsymbol{k},m\boldsymbol{q}} \langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{q}}\rangle\, c^\dagger_{n\boldsymbol{k}} c_{m\boldsymbol{q}} \tag{1.76}$$

$c^\dagger_{n\boldsymbol{k}}$: Creation operator for fermion in Bloch state $|\Psi_{n\boldsymbol{k}}\rangle$

Remember that this recipe produces an operator on Fock space that acts like the single-particle operator J_i within the one-fermion subspace.

6 | Eq. (1.75) → [Here nk' is short for $(nk)' = n'k'$.]

$$\sum_{n\neq 0} \frac{\langle \mathbf{0}|\mathfrak{J}_y|\mathbf{n}\rangle\langle \mathbf{n}|\mathfrak{J}_x|\mathbf{0}\rangle}{(E_{\mathbf{n}}-E_{\mathbf{0}})^2} = \sum_{nk',mq'}\sum_{nk,mq} \langle \Psi_{nk}|J_y|\Psi_{mq}\rangle\langle \Psi_{nk'}|J_x|\Psi_{mq'}\rangle \underbrace{\sum_{\mathbf{n}\neq\mathbf{0}} \frac{\langle \mathbf{0}|c^\dagger_{nk}c_{mq}|\mathbf{n}\rangle\langle \mathbf{n}|c^\dagger_{nk'}c_{mq'}|\mathbf{0}\rangle}{(E_{\mathbf{n}}-E_{\mathbf{0}})^2}}_{\frac{\delta_{nk=mq'}\delta_{mq=nk'}\delta_{\varepsilon_m(q)>E_F}\delta_{\varepsilon_n(k)<E_F}}{[\varepsilon_m(q)-\varepsilon_n(k)]^2}} \tag{1.77}$$

$$\stackrel{\circ}{=} \sum_{\substack{nk,mq\\ \varepsilon_n(k)<E_F<\varepsilon_m(q)}} \frac{\langle \Psi_{nk}|J_y|\Psi_{mq}\rangle\langle \Psi_{mq}|J_x|\Psi_{nk}\rangle}{[\varepsilon_m(q)-\varepsilon_n(k)]^2} \tag{1.78}$$

To evaluate the sum $\sum_{\mathbf{n}\neq\mathbf{0}}$ over all excited many-body states, convince yourself that you can *w.l.o.g.* replace the denominator by $[\varepsilon_m(q)-\varepsilon_n(k)]^2$ (which is independent of $\mathbf{n}$!). Then $\sum_{\mathbf{n}\neq\mathbf{0}} |\mathbf{n}\rangle\langle\mathbf{n}|$ can be written as $\mathbb{1} - |\mathbf{0}\rangle\langle\mathbf{0}|$ and the rest follows.

7 | Assume $\varepsilon_n(k) \lessgtr E_F$ for all $k \in T^2$

¡! This means that the Fermi energy falls into a *band gap*. This is absolutely crucial for what follows.

(Note that statements like "$\varepsilon_n < E_F$" are now well-defined since $\varepsilon_n(k) < E_F$ is true for all momenta and only depends on the band index n.)

→

$$\sigma_{xy} \stackrel{\circ}{=} \frac{i\hbar}{A} \sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}} \sum_{k,q\in T^2} \frac{\begin{Bmatrix} \langle \Psi_{nk}|J_y|\Psi_{mq}\rangle\langle \Psi_{mq}|J_x|\Psi_{nk}\rangle \\ -\langle \Psi_{nk}|J_x|\Psi_{mq}\rangle\langle \Psi_{mq}|J_y|\Psi_{nk}\rangle \end{Bmatrix}}{[\varepsilon_m(q)-\varepsilon_n(k)]^2} \tag{1.79}$$

8 | As a first simplification, we want to get rid of one of the two momentum summations. To do so, we must show that the current operator cannot change the momentum of a state:

i | Define the single-particle current operator

$$J := e\frac{i}{\hbar}[H_0, x] \tag{1.80}$$

This definition is motivated as follows: Physically, a sensible *single particle* current operator must satisfy $\langle J\rangle = e\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t}$ = Charge × Velocity. The ↓ *Ehrenfest theorem* tells us that $\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\langle[H_0,x]\rangle$ which immediately suggests the definition (1.80). You can easily check that for a free particle, $H_0 = \frac{p^2}{2m}$, it is $J = e\frac{p}{m}$ (as it should be).

ii | ∢ Translation operator T_R with lattice vector R:

$$T_R x T_R^{-1} = x + R \tag{1.81a}$$

$$T_R H_0 T_R^{-1} = H_0 \tag{1.81b}$$

$$T_R|\Psi_{nk}\rangle = e^{ikR}|\Psi_{nk}\rangle \tag{1.81c}$$

- The first equation follows from the definition of the translation operator.
- The commutativity with the Hamiltonian follows from our assumption that the system features a discrete translation invariance ("periodic potential").

- The energy eigenstates of such a Hamiltonian are Bloch states $|\Psi_{n\boldsymbol{k}}\rangle$ which are also eigenstates of these lattice translations (this is just the statement of ← *Bloch's theorem*).

iii | Consequently

$$T_{\boldsymbol{R}}\boldsymbol{J}T_{\boldsymbol{R}}^{-1} = i\frac{e}{\hbar}[H_0, \boldsymbol{x}+\boldsymbol{R}] = i\frac{e}{\hbar}[H_0, \boldsymbol{x}] = \boldsymbol{J} \tag{1.82}$$

→ $\boldsymbol{J}$ cannot change lattice momenta

Formally: $\langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{q}}\rangle = \langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{k}}\rangle\delta_{\boldsymbol{k},\boldsymbol{q}}$

iv | Thus Eq. (1.79) →

$$\sigma_{xy} \mathrel{\hat{=}} \frac{i\hbar}{A}\sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}}\sum_{\boldsymbol{k}\in T^2}\frac{\begin{Bmatrix}\langle\Psi_{n\boldsymbol{k}}|J_y|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_x|\Psi_{n\boldsymbol{k}}\rangle\\ -\langle\Psi_{n\boldsymbol{k}}|J_x|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_y|\Psi_{n\boldsymbol{k}}\rangle\end{Bmatrix}}{[\varepsilon_m(\boldsymbol{k})-\varepsilon_n(\boldsymbol{k})]^2} \tag{1.83}$$

9 | ⊲ Thermodynamic limit (in real space): $L_i \to \infty$

⇔ Continuum limit (in momentum space): $\Delta k_i \equiv \frac{2\pi}{L_i} \to 0$

→ The sum over momenta turns into an integral over the Brillouin zone T^2:

$$\sigma_{xy} \mathrel{\hat{=}} i\hbar\sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}}\int_{T^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\begin{Bmatrix}\langle\Psi_{n\boldsymbol{k}}|J_y|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_x|\Psi_{n\boldsymbol{k}}\rangle\\ -\langle\Psi_{n\boldsymbol{k}}|J_x|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_y|\Psi_{n\boldsymbol{k}}\rangle\end{Bmatrix}}{[\varepsilon_m(\boldsymbol{k})-\varepsilon_n(\boldsymbol{k})]^2} \tag{1.84}$$

- The continuum limit is convenient because we can now use tools from calculus to simplify this expression further.
- Here we used the usual approximation of a Riemann sum:

$$\frac{1}{L_i}\sum_{k_i} = \frac{1}{2\pi}\sum_{k_i}\frac{2\pi}{L_i} \xrightarrow{L_i\to\infty} \int\frac{\mathrm{d}k_i}{2\pi} \tag{1.85}$$

Remember that $A = L_xL_y$.

10 | Our next goal is to get rid of the current operators:

i | Use $|\Psi_{n\boldsymbol{k}}\rangle = e^{i\boldsymbol{k}\boldsymbol{x}}|u_{n\boldsymbol{k}}\rangle$ (← *Bloch theorem*) and define $\tilde{\boldsymbol{J}}(\boldsymbol{k}) := e^{-i\boldsymbol{k}\boldsymbol{x}}\boldsymbol{J}e^{i\boldsymbol{k}\boldsymbol{x}}$ so that

$$\langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{k}}\rangle = \langle u_{n\boldsymbol{k}}|\tilde{J}_i(\boldsymbol{k})|u_{m\boldsymbol{k}}\rangle \tag{1.86}$$

¡! Note that in $e^{i\boldsymbol{k}\boldsymbol{x}}$, $\boldsymbol{x}$ is the position *operator*.

ii | Define $\tilde{H}_0(\boldsymbol{k}) := e^{-i\boldsymbol{k}\boldsymbol{x}}H_0e^{i\boldsymbol{k}\boldsymbol{x}}$ so that

$$H_0|\Psi_{n\boldsymbol{k}}\rangle = \varepsilon_n(\boldsymbol{k})|\Psi_{n\boldsymbol{k}}\rangle \quad\Leftrightarrow\quad \tilde{H}_0(\boldsymbol{k})|u_{n\boldsymbol{k}}\rangle = \varepsilon_n(\boldsymbol{k})|u_{n\boldsymbol{k}}\rangle \tag{1.87}$$

iii | With these preliminaries, we can write:

$$\tilde{J}_i \mathrel{\hat{=}} \frac{e}{\hbar}\tilde{\partial}_i\tilde{H}_0 \quad\text{with}\quad \tilde{\partial}_i := \frac{\partial}{\partial k_i} \tag{1.88}$$

To show this use the definition of $\tilde{H}_0(\boldsymbol{k})$ and show that $\tilde{\partial}_i\tilde{H}_0 = i[\tilde{H}_0, x]$.

iv | Eqs. (1.84), (1.86) and (1.88) →

$$\sigma_{xy} \mathrel{\hat{=}} i\frac{e^2}{\hbar}\sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}}\int_{T^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\left\{\begin{array}{c}\langle u_{n\boldsymbol{k}}|\tilde\partial_y\tilde H_0|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}|\tilde\partial_x\tilde H_0|u_{n\boldsymbol{k}}\rangle\\ -\langle u_{n\boldsymbol{k}}|\tilde\partial_x\tilde H_0|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}|\tilde\partial_y\tilde H_0|u_{n\boldsymbol{k}}\rangle\end{array}\right\}}{[\varepsilon_m(\boldsymbol{k})-\varepsilon_n(\boldsymbol{k})]^2} \tag{1.89}$$

11 | Use

$$\langle u_{n\boldsymbol{k}}|\tilde\partial_y\tilde H_0|u_{m\boldsymbol{k}}\rangle = \langle u_{n\boldsymbol{k}}|\tilde\partial_y\left(\tilde H_0|u_{m\boldsymbol{k}}\rangle\right) - \langle u_{n\boldsymbol{k}}|\tilde H_0|\tilde\partial_y u_{m\boldsymbol{k}}\rangle \tag{1.90a}$$

$$= [\varepsilon_m(\boldsymbol{k})-\varepsilon_n(\boldsymbol{k})]\langle u_{n\boldsymbol{k}}|\tilde\partial_y u_{m\boldsymbol{k}}\rangle \tag{1.90b}$$

$$= [\varepsilon_n(\boldsymbol{k})-\varepsilon_m(\boldsymbol{k})]\langle\tilde\partial_y u_{n\boldsymbol{k}}|u_{m\boldsymbol{k}}\rangle \tag{1.90c}$$

The first line is just the product rule, in the second line we used that $\tilde H_0 = \tilde H_0^\dagger$ and that $\langle u_{n\boldsymbol{k}}|u_{m\boldsymbol{k}}\rangle = 0$ for $n\neq m$ (which is the case in our expression for the Hall conductivity). The last line follows if in the first line the derivative acts on the bra to the left instead on the ket to the right.

→

$$\sigma_{xy} \mathrel{\hat{=}} i\frac{e^2}{\hbar}\sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}}\int_{T^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\begin{array}{c}\langle\tilde\partial_y u_{n\boldsymbol{k}}|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}|\tilde\partial_x u_{n\boldsymbol{k}}\rangle\\ -\langle\tilde\partial_x u_{n\boldsymbol{k}}|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}|\tilde\partial_y u_{n\boldsymbol{k}}\rangle\end{array}\right\} \tag{1.91}$$

Yay! The denominator is gone ...☺

12 | Use

$$\sum_m |u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}| = \mathbb{1} \tag{1.92a}$$

$$\Rightarrow \sum_{m:\varepsilon_m>E_F}|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}| = \mathbb{1} - \sum_{m:\varepsilon_m<E_F}|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}| \tag{1.92b}$$

These statements are true on the subspace spanned by the Bloch functions $|u_{n\boldsymbol{k}}\rangle$ for fixed $\boldsymbol{k}$.

More rigorously, one should replace $\mathbb{1}$ by the projector $P_{\boldsymbol{k}}$ onto states with lattice momentum $\boldsymbol{k}$ and do the derivatives in the expression for σ_{xy} properly; the result will be the same, though.

→

$$\boxed{\sigma_{xy} \mathrel{\hat{=}} i\frac{e^2}{\hbar}\sum_{n:\varepsilon_n<E_F}\int_{T^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\langle\tilde\partial_y u_{n\boldsymbol{k}}|\tilde\partial_x u_{n\boldsymbol{k}}\rangle - \langle\tilde\partial_x u_{n\boldsymbol{k}}|\tilde\partial_y u_{n\boldsymbol{k}}\rangle\right\}} \tag{1.93}$$

Only the term with $\mathbb{1}$ survives. The second term vanishes as it replaces the sum over empty bands by a sum over filled bands. But then the sum in the expression for the Hall conductance vanishes identically if one shifts the derivatives to the states with $m\boldsymbol{k}$ in the first term [using Eq. (1.90)] and substitutes $n\leftrightarrow m$ in the sums (the last step only works because m and n now run over the same range of filled bands).